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s-NUMBERS OF PROJECTIONS IN BANACH SPACES

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ABSTRACT

Given an s-number sequence $s \in \{h, x, y, c, d, a, \Gamma\}$, we find a characterization of the following property of a Banach space X: (P_s). There is a constant $C > 0$ such that, for any n-dimensional subspace E of X, we can find a projection P from X onto E with $\sup_k ks_k(P) \leqq Cn$. As an application, we prove that X has weak type 2 if and only if X is finite dimensionally norming, thus answering a question of Casazza and Shura. Weak Hilbert spaces are also characterized in a new way, the main tool in the proof being a characterization of weak cotype 2 by means of projections. The latter is applied to the study of U.A.P., too.

§0. Introduction

A fundamental result due to Lindenstrauss and Tzafriri [5] states that a Banach space X is isomorphic to a Hilbert space if and only if there is a constant $C > 0$ such that, for every finite-dimensional subspace, there is a projection P onto it with $\| P \| \leqq C$. Recently, Pisier has proved a K-convex version of this theorem ([13], 11.15): X is K-convex if and only if there is a constant $C > 0$ such that, for every n-dimensional subspace, we can find a projection P onto it with $e_n(P) \leqq C$ (where $e_n(P)$ is the n-th entropy number of P: see Section 1 for the definitions). In this paper we investigate the properties defined replacing the entropy numbers with different s-numbers. We are able to find corresponding characterizations of these properties for the Hilbert (h_k), Weyl (x_k), Chang (y_k), Gelfand (c_k), Kolmogorov (d_k), approximation (a_k) and Grothendieck (Γ_k) numbers. More precisely, let $s \in \{h, x, y, c, d, a, e, \Gamma\}$ and define the property

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(P_s) *There exists a constant* $C > 0$ *such that, for all n-dimensional subspaces* E *of* X, *there is a projection from* X *onto* E *with*

$$\sup_k ks_k(P) \leqq Cn.$$

With this notation, Pisier's result reads: *X is K-convex iff X has* (P_e). Concerning the other s-numbers, we are going to prove that

(1) *every Banach space has* (P_h) *and* (P_x),
(2) *X has* (P_y) *iff X has* (P_d) *iff X has weak type* 2,
(3) *X has* (P_c) *iff* X^* *has weak type* 2,
(4) *X has* (P_a) *iff X has* (P_Γ) *iff X is a weak Hilbert space.*

We prove this in Section 3 (Theorems 5–8) together with some refinements, one of which is the following weak Hilbert space version of the Lindenstrauss–Tzafriri theorem cited above: X is a weak Hilbert space iff there exist constants $C > 0$ and $\delta \in (0, \frac{1}{2})$ such that, for every n-dimensional subspace E of X, there is a projection P from X onto E with $a_{[\delta n]}(P) \leqq C$.

In Section 4, Theorem 6 is applied to prove that X has weak type 2 iff X is "finite dimensionally norming". The latter property has been introduced by Casazza and Shura and, at first, it has been supposed to be stronger than weak type 2 (see [2, Appendix]).

An important tool in our proofs is provided in Section 2. Here we give a new characterization of weak cotype 2 by means of projections (Theorem 1). As an application, we prove that a local form of the U.A.P. implies weak cotype 2 (Corollary 3).

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§1. Notation

$X, Y, \ldots, E, F, \ldots$ will be Banach spaces, the letters $E, F, \ldots$ being usually reserved for finite-dimensional spaces. $\mathrm{Dim}(X)$ (resp. $\mathrm{Dim}_n(X)$) is the class of all finite-dimensional (resp. of all n-dimensional) subspaces of X. B_X is the closed unit ball and X^* is the dual of X.

Let $u: X \to Y$ be an operator ($=$ continuous linear map) and let $k \in \mathbf{N}$. Following Pietsch [10, 11] we define the k-th *approximation* (resp.

Weyl, Chang, Hilbert, Gelfand, Kolmogorov, entropy, Grothendieck) number of u by

$$a_k(u) := \inf\{\, \| u - v \| : v : X \to Y, \operatorname{rank}(v) < k \},$$

$$x_k(u) := \sup\{a_k(uv) : v : l_2 \to X, \| v \| \leqq 1\},$$

$$y_k(u) := \sup\{a_k(vu) : v : Y \to l_2, \| v \| \leqq 1\},$$

$$h_k(u) := \sup\{a_k(vuw) : w : l_2 \to X, v : Y \to l_2, \ \| w \| \leqq 1, \ \| v \| \leqq 1\},$$

$$c_k(u) := \inf\{\, \| u \,|_Z \| : Z \subset X, \operatorname{codim} Z < k\},$$

$$d_k(u) := \inf\{\, \| q_W u \| : W \subset Y, \dim W < k, q_W : Y \to Y/W \text{ the quotient map}\},$$

$$e_k(u) := \inf\left\{\varepsilon > 0 : \exists\, y_1, \ldots, y_{2^{k-1}} \in Y \text{ with } u(B_X) \subset \bigcup_{i=1}^{2^{k-1}} (y_i + \varepsilon B_Y)\right\},$$

$$\Gamma_k(\omega) := \sup\{\, | \det[(ux_i, z_j)] |^{1/k} : x_1, \ldots, x_k \in B_X, z_1, \ldots, z_k \in B_{Y^*}\}.$$

The main properties of these s-numbers may be found in [1, 3, 10, 11].

Given $u : X \to Y$, we denote by $\| u \|^{(s)}_{q,\infty} := \sup_k k^{1/q} s_k(u)$, where $1 \leqq q < \infty$ and s_k is any s-number sequence.

u is said to be 2-summing if there is a constant $c > 0$ such that, for all finite sequences $x_1, \ldots, x_n$ in X,

$$\left(\sum_{i=1}^{n} \| ux_i \|^2 \right)^{1/2} \leqq c \cdot \sup_{z \in B_{X^*}} \left(\sum_{i=1}^{n} \langle z, x_i \rangle^2 \right)^{1/2}.$$

In this case we let $\pi_2(u) = \inf c$.

For any operator $u : l_2^n \to X$, the $l(u)$ norm is defined by

$$l(u) := \left(\int \| ux \|^2 \gamma_n(dx) \right)^{1/2},$$

where γ_n is the canonical Gaussian measure on $\mathbf{R}^n$. Further, for any $v : X \to l_2^n$, we let

$$l^*(v) := \sup\{\, | \operatorname{tr}(uv) | : u : l_2^n \to X, l(u) \leqq 1\}.$$

The following lemma is a standard consequence of a theorem by Lewis and known properties of the norms π_2, l and l^*:

Lemma 0. *Let $E \in \operatorname{Dim}_n(X)$. Then there exist operators*

$$u_E : l_2^n \to E, \qquad v_E : X \to l_2^n$$

and

$$w_E : l_2^n \to E, \qquad z_E : X \to l_2^n$$

such that $v_E u_E = z_E w_E = \mathrm{id}_{l_2^n}$ *and* $\| u_E \| = 1$, $\pi_2(v_E) = l(w_E) = l^*(z_E) = n^{1/2}$.

Given $E \in \mathrm{Dim}_n(X)$, in the proofs to follow we shall always refer to u_E, v_E, w_E, z_E as to the operators given by the above lemma.

As for the definitions of K-convexity, weak cotype 2, weak type 2 and of weak Hilbert spaces, we refer to Pisier's forthcoming work [13].

§2. A characterization of weak cotype 2

By Definition [7], a Banach space X has weak cotype 2 if, for any $\delta \in (0, 1)$, there exists a constant $C = C(\delta) > 0$ such that, for every n-dimensional subspace E of X, we can find a subspace F of E with $\dim F \geqq \delta n$ and $d_F \leqq C$, where $d_F := d(F, l_2^{\dim F})$ is the Banach–Mazur distance between F and the Hilbert space of corresponding dimension. So, it is difficult to predict at a glance what sort of projections we are able to find in weak cotype 2 spaces. Further, it is by no means clear (from the definition) if one can avoid speaking about the l_2^n spaces in defining this concept. Now, clarifying (at least a bit) the relationship between weak cotype 2 and projections and avoiding citations of l_2^n spaces are both achieved by Theorem 1:

THEOREM 1. *For a Banach space X the following are equivalent*:

(a) *X has weak cotype* 2.

(b) *For every* $\alpha > 1$, $\varepsilon_1, \varepsilon_2 > 0$ *such that* $\varepsilon_1 + \varepsilon_2 < 1$, *there is a constant* $C = C(\alpha, \varepsilon_1, \varepsilon_2) > 0$ *such that the following holds*: *for all subspaces* $E \subset F$ *of* X *with* $\dim E = n$ *and* $\dim F \leqq \alpha n$, *we can find a subspace* Z *of* F *with* $\operatorname{codim} Z \leqq \varepsilon_1 n$, *a subspace* E_0 *of* E *with* $\dim E_0 \geqq (1 - \varepsilon_2)n$ *and a projection* P *from* Z *onto* $E_0 \cap Z$ *such that* $\| P \| \leqq C$.

(c) *There exist* $\varepsilon_1, \varepsilon_2 > 0$ *with* $\varepsilon_1 + \varepsilon_2 < \frac{1}{2}$ *such that the statement of* (b) *holds with* $\alpha = 2$.

PROOF. (a)$\Rightarrow$(b). Let X have weak cotype 2 and fix $\alpha > 1$, $\varepsilon_1, \varepsilon_2 > 0$ such that $\varepsilon_1 + \varepsilon_2 < 1$, and $E \subset F \subset X$ with $\dim E = n$, $\dim F \leqq \alpha n$. By definition of weak cotype 2, there exists a constant $C = C(\varepsilon_1) > 0$ and a subspace Z of F such that $\operatorname{codim} Z \leqq \varepsilon_1 n$ and $d_Z \leqq C$. The projection P from Z onto

$Z \cap E$ is now easy to produce since Z is C-euclidean. Taking $E_0 := E$ we see that (b) holds.

(b)$\Rightarrow$(c) is trivial.

(c)$\Rightarrow$(a). We will use a variation of the argument of [12], pp. 561–562, which is itself a variation of the main argument of [5]. Let X satisfy (c) and fix E in X with $\dim E = n$. Using Dvoretzky's Theorem and a classical argument, it is not hard to see that $E \bigoplus_2 l_2^n$ can be C'-embedded into X, where C' is a uniform constant. Let $T: E \to l_2^n$ be an isomorphism with $\| T \| = \| T^{-1} \| = (d_E)^{1/2}$. By (c), there are subspaces Z of $E \bigoplus_2 l_2^n$, G of $D := \{(e, Te) : e \in E\}$, and a projection P from Z onto $Z \cap G$ such that $\operatorname{codim} Z \leqq \varepsilon_1 n$, $\dim G \geqq (1 - \varepsilon_2)n$ and $\| P \| \leqq C'C$.

Let $F \subset E$ be such that $G = \{(f, Tf) : f \in F\}$ and define

$$F_1 := \{ f \in F : (f, 0) \in Z\},$$

$$F_2 := \{ f \in F : (0, Tf) \in Z\},$$

$$F_0 := F_1 \cap F_2.$$

Clearly, $\dim F_i \geqq (1 - (\varepsilon_1 + \varepsilon_2))n$, $i = 1, 2$, and so $\dim F_0 \geqq (1 - 2(\varepsilon_1 + \varepsilon_2))n$.

Define the operators $\alpha : E \to E$, $\beta : l_2^n \to E$ by

$$P(x, y) = (\alpha(x) + \beta(y), T\alpha(x) + T\beta(y)), \qquad \forall (x, y) \in E \bigoplus_2 l_2^n.$$

Then, by the choice of F_0,

$$\max\{ \| T\alpha |_{F_0} \|, \| \beta |_{TF_0} \| \} \leqq \| P |_Z \| \leqq C'C.$$

Since P is a projection, we have $\mathrm{id}_{F_0} = T^{-1}(T\alpha |_{F_0}) + \beta |_{TF_0} T |_{F_0}$, hence

$$\begin{aligned} d_{F_0} &= \gamma_2(\mathrm{id}_{F_0}) \\ &\leqq \gamma_2(T^{-1}(T\alpha |_{F_0})) = \gamma_2(\beta |_{TF_0} T |_{F_0}) \\ &\leqq \| T^{-1} \| \, \| T\alpha |_{F_0} \| + \| \beta |_{TF_0} \| \, \| T \| \\ &\leqq 2C'C(d_E)^{1/2}. \end{aligned}$$

Now, by the iteration method of Milman (see, e.g., [6]) and by [7] Theorem 1 this inequality is known to imply that X has weak cotype 2 (remember that E was arbitrary and that F_0 has dimension proportional to n). □

With only minor changes in the proof we get another statement if we introduce the Gelfand numbers:

THEOREM 2. *For a Banach space X the following are equivalent:*

(a) *X has weak cotype 2.*

(b) *For every $\alpha > 1$ and $\varepsilon \in (0, 1)$ there is a constant $C = C(\alpha, \varepsilon) > 0$ such that the following holds: for all subspace $E \subset F$ of X with* $\dim E = n$ *and* $\dim F \leqq \alpha n$ *there exists a projection P from F onto E with $c_{[\varepsilon n]}(P) \leqq C$.*

(c) *There is an $\varepsilon \in (0, \frac{1}{2})$ such that the statement of* (b) *holds with $\alpha = 2$.*

The rest of this section will not be used in the sequel.

The next corollary deals with what may be called a local version of U.A.P.:

COROLLARY 3. *Suppose there exist constants $\varepsilon \in (0, \frac{1}{2})$ and $C > 0$ such that, for all subspaces $E \subset F$ of X with* $\dim E = n = \frac{1}{2}\dim F$, *we can find an operator $T: F \to X$ such that*

(i) $\| T \| \leqq C$,

(ii) *$Te = e$ for all $e \in E$,*

(iii) rank $T \leqq (1 + \varepsilon)n$.

Then X has weak cotype 2.

PROOF. Let $E \subset F \subset X$, $\dim E = n = \frac{1}{2}\dim F$, and let $T: F \to X$ satisfy (i)–(iii). Let P_0 be a projection from $T(F)$ onto E and define $P := P_0 T$. Then P is a projection from F onto E and

$$c_{[\varepsilon n]}(P) \leqq \| T \| \, c_{[\varepsilon n]}(P_0) \leqq C \, \| P_0 |_E \| = C,$$

Since the codimension of E in $T(F)$ is less than εn. Since $\varepsilon < \frac{1}{2}$, we see that X satisfies (c) in Theorem 2, so X has weak cotype 2. □

REMARKS. (i) Notice that if we take $\varepsilon = 0$ in Corollary 3 then the hypothesis actually implies by [5] that X is isomorphic to a Hilbert space.

(ii) If we strengthen the hypothesis of Corollary 3 so as to require that the constants ε and C be small enough, then X must also be K-convex.

As an illustration, we give the following

FACT. *Let X satisfy the hypothesis of Corollary 3 and assume that*

$$C\varepsilon^{1/2}(1 + \varepsilon)^{-1/2} < \pi^{3/2} 2^{-9/2} e^{-1} = 0.09\ldots.$$

Then X^* has weak type 2.

PROOF. We have only to prove that X is K-convex. Assume the contrary, and note that w.l.o.g. we may suppose that X contains isometrical copies of l_1^n. Let $F = l_1^{2n}$, $E \subset F$ such that $\dim E = n$ and $d_E \leqq 16e\pi^{-1}$ (see e.g. [13], Theorem 6.1). Let $T: F \to X$ satisfy (i)–(iii). Then, if P is a projection from $T(F)$ onto E with $\| P \| \leqq (1+\varepsilon)^{-1} + (\varepsilon n)^{1/2}(1+\varepsilon)^{-1/2}$ ([4], p. 237), we get

$$\gamma_1(l_2^n) \leqq d_E \| T \| \| P \| \leqq \frac{16e}{\pi} C((1+\varepsilon)^{-1} + (\varepsilon n)^{1/2}(1+\varepsilon)^{-1/2}).$$

Since $\lim_{n\to\infty} n^{-1/2}\gamma_1(l_2^n) = (\pi/2)^{1/2}$ ([10], 28.2.6), we obtain the inequality $\pi^{3/2}2^{-9/2}e^{-1} \leqq C\varepsilon^{1/2}(1+\varepsilon)^{-1/2}$, which contradicts our assumption on ε and C. □

In [12], §5, Pisier asks what can be said about the relationship between the U.A.P. with uniformity function $g(n) \leqq cn$ and the weak Hilbert space property. The question is among others motivated by the fact that U.A.P. with $g(n) = n$ characterizes Hilbertian spaces ([5]). A partial step in this direction is the next immediate consequence of Corollary 3:

Corollary 4. *If X has the U.A.P. with uniformity function $g(n) \leqq cn$, where $c < 3/2$, then X has weak cotype 2.*

Remarks. (i) Of course, remark (ii) following Corollary 3 holds in this case, too.

(ii) Using different methods, I was able to prove Corollary 4 without the restriction $c < \frac{3}{2}$.

§3. The main theorems

Theorem 5. *For every Banach space X and any $E \in \mathrm{Dim}_n(X)$ there exists a projection $P: X \to E$ such that*

$$x_k(P) \leqq (n/k)^{1/2}, \qquad k \in \mathbf{N}.$$

Consequently, every Banach space enjoys (P_h) *and* (P_x).

Proof. Let $E \in \mathrm{Dim}_n(X)$ and define $P := u_E v_E$ (Lemma 0). Then, by [11], 2.7.3,

$$\| P \|_{2,\infty}^{(x)} \leqq \pi_2(P) \leqq \| u_E \| \pi_2(v_E) = n^{1/2},$$

which proves the first assertion. Property (P_x) follows trivially and (P_h) follows from (P_x) because of the easy inequality $h_k \leqq x_k$. □

Theorem 6. *The following are equivalent:*

(i) *X has* (P_d).

(ii) *X has* (P_y).

(iii) *There exist constants* $C>0$ *and* $\delta\in(0,1)$ *such that, for any* $E\in \mathrm{Dim}_n(X)$, *there is a projection* $P: X\to E$ *with* $y_{[\delta n]}(P)\leqq C$.

(iv) *X has weak type* 2.

Proof. (i)$\Rightarrow$(ii) follows from the inequality $y_k\leqq d_k$ ([11], 2.5.13), and (ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(iv). Let C and δ be as in (iii). Fix $E\in\mathrm{Dim}_n(X)$, let $P: X\to E$ be a projection with $y_{[\delta n]}(P)\leqq C$, and let $v: E\to l_2^n$ be an operator. Since $d_{[\delta n]}(vP)=y_{[\delta n]}(vP)\leqq C\,\|v\|$, by the definition of the Kolmogorov numbers there is an orthogonal projection $q: l_2^n\to l_2^k$, where $k\geqq[(1-\delta)n]$, such that $\|qvP\|\leqq C\,\|v\|$. Since obviously qvP extends qv, we may proceed as in the proof of Theorem 10 of [7] to deduce that X^* has weak cotype 2 and that X is locally π-euclidean (i.e., K-convex).

(iv)$\Rightarrow$(i). Let X have weak type 2, $E\in\mathrm{Dim}_n(X)$ and $P:=w_E z_E$ (lemma 0). Then, by [10], 11.8.2,

$$d_k(P)\leqq d_{[k/2]}(w_E)d_{[k/2]}(z_E)=d_{[k/2]}(w_E)a_{[k/2]}(z_E).$$

Now, since $d_{[k/2]}(w_E)=c_{[k/2]}(w_E')$ ([10], 11.7.6), by [8], Proposition 1, there is an absolute constant $\kappa>0$ such that

$$d_{[k/2]}(w_E)\leqq\kappa[k/2]^{-1/2}l(w_E).$$

Further, by the definition of weak type 2,

$$a_{[k/2]}(z_E)\leqq wT_2(X)[k/2]^{-1/2}l^*(z_E).$$

Combining all this and using the lemma we get a constant $C=C(X)$ such that

$$d_k(P)\leqq C\frac{n}{k},$$

which shows that X has (P_d). □

The characterization of (P_c) appears to the the most deep one. In view of the application to weak Hilbert spaces (see Theorem 8), it may be interesting to note that an important step in the proof of the next theorem is achieved by a variation of the original Lindenstrauss–Tzafriri argument in [5] (see Theorem 1).

Theorem 7. *X has* (P_c) *iff* X^* *has weak type* 2, *i.e., iff X is K-convex and has weak cotype* 2.

Proof.

Step 1. *If* X^* *has weak type* 2, *then X has* (P_c).

Let $E \in \mathrm{Dim}_n(X)$ and $P := w_E z_E$. By Proposition 1 of [8], there is $\kappa > 0$ such that

$$\| z_E \|_{2,\infty}^{(c)} \leqq \kappa l(z_E^*) \leqq \kappa K(X) l^*(z_E) = \kappa K(X) n^{1/2}$$

(here $K(X)$ is the K-convexity constant of X). On the other hand, since X has weak cotype 2, we have

$$\| w_E \|_{2,\infty}^{(c)} \leqq wC_2(X) l(w_E) = wC_2(X) n^{1/2}.$$

Finally, by [11], 2.4.9, we get a constant $\kappa' > 0$ such that

$$\| P \|_{1,\infty}^{(c)} \leqq \kappa' \| w_E \|_{2,\infty}^{(c)} \| Z_E \|_{2,\infty}^{(c)} \leqq [\kappa' \kappa K(X) wC_2(X)] n,$$

hence X has (P_c).

Step 2. *If X has* (P_c) *then X is K-convex*.

This follows at once from a result by Carl [1, Theorem 1], which states that there is a constant κ such that $\| \cdot \|_{1,\infty}^{(e)} \leqq \kappa \| \cdot \|_{1,\infty}^{(c)}$, and from Pisier's characterization of K-convexity by means of (P_e) ([13], 11.15).

Step 3. *If X has* (P_c) *then X has weak cotype* 2.

Let X have (P_c) and let $\delta \in (0, \frac{1}{2})$. Then, for any $E \in \mathrm{Dim}_n(X)$ there exists a projection P from X onto E with $c_{[\delta n]}(P) \leqq C/\delta$. By Theorem 2, we get that X has weak cotype 2. □

Theorem 8. *The following are equivalent*:

(i) *X has* (P_a).

(ii) *There exist constants* $C > 0$ *and* $\delta \in (0, \frac{1}{2})$ *such that, for any* $E \in \mathrm{Dim}_n(X)$, *there is a projection P from X onto E with* $a_{[\delta n]}(P) \leqq C$.

(iii) *X has* (P_Γ).

(iv) *There exists a constant* $C > 0$ *such that, for any* $E \in \mathrm{Dim}_n(X)$, *there is a projection* $P : X \to E$ *with* $\Gamma_n(P) \leqq C$.

(v) *X is a weak Hilbert space*.

(vi) *There exists a constant* $C > 0$ *such that, for any* $E \in \mathrm{Dim}_n(X)$, *there is a projection* $P : X \to E$ *with* $\| P \|_{2,\infty}^{(a)} \leqq Cn^{1/2}$.

Remark. The equivalence between (ii) and (v) may be regarded as a weak Hilbert space version of the Lindenstrauss–Tzafriri result in [5].

Proof. The implications (v)⇒(i)⇒(ii) and (iii)⇒(iv) are trivial.

(ii)$\Rightarrow$(v). If X satisfies (ii), then by the inequalities $c_k \leqq a_k$ and $y_k \leqq a_k$ ([11], 2.3.4), X satisfies condition (iii) of Theorem 6 and condition (c) in Theorem 2 as well: these facts together imply that X has weak type 2 and weak cotype 2, i.e. X is a weak Hilbert space.

(v)$\Rightarrow$(vi). Let $E \in \mathrm{Dim}_n(X)$, $P := u_E v_E$ (Lemma 0). Then (by [11], 2.3.16)

$$\| P \|_{2,\infty}^{(a)} = \| v_E^* u_E^* \|_{2,\infty}^{(a)} \leqq \| v_E^* \|_{2,\infty}^{(a)} \leqq C\pi_2(v_E) = Cn^{1/2},$$

where $C = C(X)$, since if X is a weak Hilbert space, then so is X^*, too.

(iv)$\Rightarrow$(v). (iv) easily implies that $\Gamma_n(\mathrm{id}_E) \leqq C$ for all $E \in \mathrm{Dim}_n(X)$. Since $\Gamma_n(id_X) = \sup_{E \in \mathrm{Dim}_n(X)} \Gamma_n(\mathrm{id}_E)$, we get $\sup_n \Gamma_n(\mathrm{id}_X) < \infty$, which is known to imply that X is a weak Hilbert space ([13], Theorem 12.6).

(v)$\Rightarrow$(iii). Let $E \in \mathrm{Dim}_n(X)$ and $P := w_E z_E$ (Lemma 0). Since X and X^* have both weak cotype 2, by [9], Theorem 2.6, there is a constant $C > 0$ such that

$$\Gamma_k(w_E) \leqq Ck^{-1/2} l(w_E) \quad \text{and} \quad \Gamma_k(z_E) = \Gamma_k(z_E') \leqq Ck^{-1/2} l(z_E'),$$

for all $k \in \mathbf{N}$. By [3], 1.1.2 and 1.1.4, we have then

$$\Gamma_k(P) = \Gamma_k(w_E)\Gamma_k(z_E) \leqq C^2 k^{-1} n^{1/2} l(z_E').$$

Now, since X is K-convex, we have $l(z_E') \leqq K(X) l^*(z_E)$, and so $\Gamma_k(P) \leqq C'(n/k)$ for all k, as we had to prove. □

Theorem 7 yields the following statement: *If X is K-convex, then the following are equivalent*:

(i) *X has weak cotype* 2.

(ii) *There exist constants $C > 0$ and $\delta \in (0, \frac{1}{2})$ such that, for any $E \in \mathrm{Dim}_n(X)$, there is a projection P from X onto E with $c_{[\delta n]}(P) \leqq C$.*

Can we avoid the K-convexity assumption?

§4. Finite dimensionally norming spaces

The definitions of finite dimensionally norming and of well-normed spaces have been given in an appendix to the book [2] by Casazza and Shura, where it is asked what exactly these properties are. We will show that weak type 2 is equivalent to "finite dimensionally norming" and that "weakly well normed" (our variation of "well-normed") has a lot to do with X^* having weak type 2.

Definition. (i) X is *finite dimensionally norming* (FDN) if there are constants $C > 0$ and $\delta \in (0, 1)$ such that, for any $E \in \mathrm{Dim}(X)$, there exists

$F \in \mathrm{Dim}(X^*)$ with $\dim F \geqq \delta \dim E$ such that E is C-norming over F, i.e. such that

$$\| f \| \leqq C \sup\{ |\langle f, e\rangle| : e \in B_E \} \qquad \text{for all } f \in F.$$

(ii) X is *weakly well-normed* (WWN) if for each $\delta \in (0, 1)$ there is a constant $C_\delta > 0$ such that, for any $E \in \mathrm{Dim}(X)$, there exist $F \in \mathrm{Dim}(X^*)$ and a subspace E_0 of E with

$$\dim F \leqq (1+\delta)\dim E, \qquad \dim E_0 \geqq (1-\delta)\dim E,$$

and such that F is C_δ-norming over E_0.

THEOREM 9. *The following are equivalent*:
(i) *X has weak type* 2.
(ii) *X is* FDN.

PROOF. (i)$\Rightarrow$(ii). Let $\delta \in (0, 1)$, $E \in \mathrm{Dim}_n(X)$. By Theorem 6 there are a constant $C > 0$ and a projection $P : X \to E$ such that $d_{[(1-\delta)n]}(P) \leqq C$. This means that there is a subspace E_0 of E with $\dim E_0 < (1-\delta)n$ and such that $\| qP \| \leqq C$, where $q : E \to E/E_0$ is the quotient map. Let $F := (qP)^*(E/E_0)^*$ ($F \subset X^*$). Then $\dim F \geqq \delta n$ and E is C-norming over F. In fact, let $f \in F$ and $e^* \in E^*$ be such that $P^*e^* = f$. Then we have

$$\| f \| = \| P^*e^* \| \leqq C \| e^* \| = C \sup\{ |\langle e^*, e\rangle| : e \in B_E \}$$
$$= C \sup\{ |\langle f, e\rangle| : e \in B_E \},$$

since $\langle e^*, e\rangle = \langle e^*, Pe\rangle = \langle P^*e^*, e\rangle = \langle f, e\rangle$ for all $e \in E$.

(ii)$\Rightarrow$(i). Let X be FDN, and consequently let $\delta \in (0, 1)$ and $C > 0$ be constants such that, given an n-dimensional subspace E of X, we can find $F \subset X^*$ with $\dim F \geqq \delta n$ and such that E is C-norming over F. Let then $G := \{ f|_E : f \in F \} \subset E^*$. Since E is C-norming over F, the map $F \to G$, $f \mapsto f|_E$ is invertible and its inverse R has norm $\leqq C$. Using the Hahn–Banach Theorem, extend R to an isomorphic embedding $\hat{R} : E^* \to X^*$ such that $\hat{R}^*|_X =: P$ is a projection of X onto E. Then, if $j : G \to E^*$ is the natural injection, we have

$$\| j^*P \| \leqq \| \hat{R}j \| = \| R \| \leqq C$$

and thus $d_{[(1-\delta)n]}(P) \leqq C$, since j^* is a quotient map of rank $\geqq \delta n$. By Theorem 6, X has weak type 2. $\square$

REMARK. It is clear from the proof that if X is FDN, then for *each* $\delta \in (0, 1)$ there is a constant $C = C(\delta)$ such that the same situation as in the definition of FDN holds.

Since it is open whether WWN implies K-convexity (I believe it does), regarding WWN we can only prove the following

THEOREM 10. (i) *If* X^* *has weak type* 2 *then* X *is* WWN.
(ii) *If* X *is* WWN *then* X *has weak cotype* 2.

PROOF. (i) Let X^* have weak type 2, $E \in \mathrm{Dim}_n(X)$, $\delta \in (0, 1)$. By Theorem 7, there are a constant $C > 0$ and a projection $P: X \to E$ such that $c_{[\delta n]}(P) \leqq C/\delta$, where C depends only on X. This means that there is a subspace Z of X with $\operatorname{cod} Z \geqq [\delta n]$ and $\| P|_Z \| \leqq C/\delta$. Let $E_0 := Z \subset E$ and $F := (X/(\ker P|_Z))^*$. Then $\dim E_0 \geqq (1 - \delta)n$ and $\dim F \leqq (1 + \delta)n$. Further, if $e \in E_0$ and $z \in \ker P|_Z$ we have $\| e \| = \| P(e - z) \| \leqq \| P|_Z \| \, \| e - z \|$, so that if e is considered as an element of $F^* = X/(\ker P|_Z)$ it follows that

$$\| e \|_E \leqq (C/\delta) \| e \|_{F^*} \qquad \text{for all } e \in E_0,$$

which means that X has WWN.

(ii) Let $E \in \mathrm{Dim}_n(X)$ and fix $\delta \in (0, \frac{1}{2})$. Let $E_0 \subset E$, $F \in \mathrm{Dim}(X^*)$ be such that $\dim E_0 \geqq (1 - \delta)n$, $\dim F \leqq (1 + \delta)n$, and let F be C_δ-norming over E_0. Let $q: X \to F^*$ be the natural quotient map and $G := q(E)$, $G_0 := q(E_0)$. Since F is C_δ-norming over E_0, $q|_{E_0}: E_0 \to G_0$ is invertible with inverse R such that $\| R \| \leqq C_\delta$.

Let $P: F^* \to G_0$ be a projection. Then, since $\dim F^* - \dim G_0 \leqq (1 + \delta)n - (1 - \delta)n = 2\delta n$, we have

$$c_{[2\delta n]+1}(RPq) \leqq c_{[2\delta n]+1}(RP) \leqq \| RP|_{G_0} \| = \| R \| \leqq C.$$

Now, RPq is a projection of X onto E_0 and so, by Theorem 1, choosing δ sufficiently small we see that X has weak cotype 2. □

REMARK. Arguing as in the remark after Corollary 3, we can see that if X is WWN and if there is a $\delta \in (0, \frac{1}{2})$ such that $C_\delta \delta^{1/2}(1 - \delta)^{-1/2}$ is small enough (e.g., strictly less than 0.09), then X must be K-convex.

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